FINITE PLANE STRAINS OF AN INCOMPRESSIBLE ELASTIC MATERIAL

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Many rubber-like materials can undergo finite elastic strains and, for moderate loads, can be considered incompressible. The plane deformation of an incompressible material has been examined within the framework of the nonlinear theory of elasticity by a number of authors. For example, such studies were conducted in [1-4] in different coordinates and with the use of different measures of strain and the corresponding stresses; along with exact solutions of certain problems obtained by a semiinverse method, the authors used the method of successive approximations in combination with expansion in a small parameter.

The plane deformation of an incompressible material is studied below in the coordinates of the strain state and the Almansi and Cauchy tensors are used as the measures of strain and stress. Equations in displacements (for the initial coordinates) and in stresses (including those for the Airy function) are established. The sufficient conditions for their ellipticity are obtained. For a Mooney material, we examine the form and type of equation for the Airy function with loads of different intensity. We also find two of its exact solutions containing free parameters. Exact solutions to problems concerning the loading of a curvilinear tetragon and an elliptical ring by special contour loads are given in a nonlinear formulation.

1. It is known [5] that the equilibrium of a uniform isotropic incompressible material in the nonlinear theory of elasticity can be described in the coordinates of the strain state by equilibrium equations, the incompressibility condition, the Murnagan law linking the stresses and strains, representations of the strains through the displacements, and expressions for the strain invariants; these are augmented by boundary conditions which may be assigned on the boundary of the deformed body:

$$\begin{gathered}\operatorname{div} P + \rho\mathbf{f} = 0,\ \varepsilon_1 - 2\varepsilon_2 + 4\varepsilon_3 = 0,\\ P = -qG + \rho(G - 2\varepsilon)\cdot\frac{dF_*}{d\varepsilon},\quad 2\varepsilon = \nabla\mathbf{u} + (\nabla\mathbf{u})^* - \nabla\mathbf{u}\cdot(\nabla\mathbf{u})^*,\\ \varepsilon_1 = \operatorname{tr}\varepsilon,\ 2\varepsilon_2 = (\operatorname{tr}\varepsilon)^2 - \operatorname{tr}\varepsilon^2,\ \varepsilon_3 = \det\varepsilon;\\ \mathbf{u}|_{\Sigma u} = \mathbf{h},\quad P\cdot\mathbf{n}|_{\Sigma p} = \mathbf{p},\end{gathered} \tag{1}$$

where ρ, q, F_* are the density, hydrostatic pressure, and elastic potential; ε_1, ε_2, and ε_3 are the principal strain invariants; $\mathbf{u}, \mathbf{f}, \mathbf{n}, \mathbf{h}, \mathbf{p}$ are the vectors of the displacements, body force, outward normal, and boundary displacements and stresses; G, P, and ε are the metric tensor and the tensors of the stresses (Cauchy) and strains (Almansi); Σ_u and Σ_p are those parts of the surface of the body on which the displacements and stresses have been assigned.

Let the material be subjected to the action of potential forces with the energy V and let it be under plane strain conditions. Then

$$\rho\mathbf{f} = -\nabla V,\ \varepsilon_3 = 0,\ F_* = F_*(\varepsilon_1)$$

and the Murnagan law reduces to a quasilinear connection between the stresses and strains

$$P = (F' - q)G - 2F'\varepsilon,\ F' = dF/d\varepsilon_1,\ F(\varepsilon_1) = \rho F_*(\varepsilon_1).$$

In this case, the main problem is the plane problem of elasticity. The relations for this problem are satisfied in a two-dimensional deformed region D, while the boundary conditions are assigned on its boundary L. In Cartesian coordinates x, y of the strain state, Eqs. (1) for the plane problem have the form

$$\frac{\partial(P_{xx} - V)}{\partial x} + \frac{\partial P_{xy}}{\partial y} = 0,\quad \frac{\partial P_{xy}}{\partial x} + \frac{\partial(P_{yy} - V)}{\partial y} = 0, \tag{2}$$

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$$\varepsilon_1 - 2\varepsilon_2 = 0,\ \varepsilon_1 = \varepsilon_{xx} + \varepsilon_{yy},\ \varepsilon_2 = \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2, \tag{2}$$

$$P_{xx} = -q + F'(1 - \varepsilon_{xx}),\ 1 - \varepsilon_{xx} = (1 - \partial u_x/\partial x)^2 + (\partial u_y/\partial x)^2,$$

$$P_{yy} = -q + F'(1 - \varepsilon_{yy}),\ 1 - \varepsilon_{yy} = (1 - \partial u_y/\partial y)^2 + (\partial u_x/\partial y)^2,$$

$$P_{xy} = -F'2\varepsilon_{xy},\ 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y}\left(1 - \frac{\partial u_x}{\partial x}\right) + \frac{\partial u_y}{\partial x}\left(1 - \frac{\partial u_y}{\partial y}\right),$$

$$u_x|_{L_u} = h_x(s),\ u_y|_{L_u} = h_y(s),$$

$$P_{xx}n_x + P_{xy}n_y|_{L_p} = p_x(s),\ P_{xy}n_x + P_{yy}n_y|_{L_p} = p_y(s)$$

(the Cartesian coordinates of the components of the vectors and tensors are designated by the same symbols as the quantities themselves, but with different subscripts; L_u and L_p are the parts of the boundary L on which the displacements and stresses were assigned, while s is the arc L).

2. With the assignment of only the displacements on the boundary of the region, the plane problem is conveniently formulated in displacements. The latter is obtained by excluding the stresses, strains, and hydrostatic pressure from (2) and is written in the form

$$\frac{\partial^2}{\partial x\,\partial y}\left\{F'\left[\left(1 - \frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2\right]\right\} - \frac{\partial^2}{\partial y^2}\left\{F'\left[\frac{\partial u_x}{\partial y}\left(1 - \frac{\partial u_x}{\partial x}\right) + \frac{\partial u_y}{\partial x}\left(1 - \frac{\partial u_y}{\partial y}\right)\right]\right\} =$$

$$= \frac{\partial^2}{\partial x\,\partial y}\left\{F'\left[\left(1 - \frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_x}{\partial y}\right)^2\right]\right\} - \frac{\partial^2}{\partial x^2}\left\{F'\left[\frac{\partial u_x}{\partial y}\left(1 - \frac{\partial u_x}{\partial x}\right) + \frac{\partial u_y}{\partial x}\left(1 - \frac{\partial u_y}{\partial y}\right)\right]\right\}, \tag{3}$$

$$\left(1 - \frac{\partial u_x}{\partial x}\right)\left(1 - \frac{\partial u_y}{\partial y}\right) - \frac{\partial u_x}{\partial y}\frac{\partial u_y}{\partial x} = 1,\ F' = F'(\varepsilon_1),$$

$$u_x|_L = h_x(s),\ u_y|_L = h_y(s),$$

where the invariant ε_1, transformed by means of the incompressibility condition, is represented by the expression

$$2\varepsilon_1 = -(\partial u_x/\partial x - \partial u_y/\partial y)^2 - (\partial u_x/\partial y + \partial u_y/\partial x)^2.$$

These relations do not contain the potential energy of the forces, and $\varepsilon_1 \le 0$ in them. The hydrostatic pressure is determined from (2) in the form

$$q + V = \int_{M_0}^{M}\left(\frac{\partial(q+V)}{\partial x}dx + \frac{\partial(q+V)}{\partial y}dy\right) + d,\ d = \text{const}, \tag{4}$$

where

$$\frac{\partial(q+V)}{\partial x} = \frac{\partial}{\partial x}\left\{F'\left[\left(1 - \frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2\right]\right\} - \frac{\partial}{\partial y}\left\{F'\left[\frac{\partial u_x}{\partial y}\left(1 - \frac{\partial u_x}{\partial x}\right) + \frac{\partial u_y}{\partial x}\left(1 - \frac{\partial u_y}{\partial y}\right)\right]\right\},$$

$$\frac{\partial(q+V)}{\partial y} = \frac{\partial}{\partial y}\left\{F'\left[\left(1 - \frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_x}{\partial y}\right)^2\right]\right\} - \frac{\partial}{\partial x}\left\{F'\left[\frac{\partial u_x}{\partial y}\left(1 - \frac{\partial u_x}{\partial x}\right) + \frac{\partial u_y}{\partial x}\left(1 - \frac{\partial u_y}{\partial y}\right)\right]\right\}, \tag{5}$$

M_0 and M are points of the region D. By virtue of (3) and (5), the integral in (4) is independent of the path of integration. The stress field is determined from Eqs. (2) and (4).

Problem (3), represented in expanded form, can be simplified somewhat if we change over from the displacements to the initial coordinates by means of the formulas $u_x = x - \xi(x, y)$, $u_y = y - \eta(x, y)$. It then takes the form [6]

$$\sum_{k+l=3}\left(A_{kl}\frac{\partial^3\xi}{\partial x^k \partial y^l} + B_{kl}\frac{\partial^3\eta}{\partial x^k \partial y^l}\right) + T = 0,\quad \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x} = 1, \tag{6}$$

$$\xi|_L = x(s) - h_x(s),\ \eta|_L = y(s) - h_y(s).$$

Here, we use T to denote the terms containing the lower derivatives, while the coefficients have the values

$$A_{30} = -F'\frac{\partial\xi}{\partial y} + F''C\frac{\partial\xi}{\partial x}, \quad A_{21} = [F' + F''(B-A)]\frac{\partial\xi}{\partial x} + F''C\frac{\partial\xi}{\partial y},$$
$$A_{03} = F'\frac{\partial\xi}{\partial x} - F''C\frac{\partial\xi}{\partial y}, \quad A_{12} = [-F' + F''(B-A)]\frac{\partial\xi}{\partial y} - F''C\frac{\partial\xi}{\partial x},$$
$$B_{30} = -F'\frac{\partial\eta}{\partial y} + F''C\frac{\partial\eta}{\partial x}, \quad B_{21} = [F' + F''(B-A)]\frac{\partial\eta}{\partial x} + F''C\frac{\partial\eta}{\partial y},$$
$$B_{03} = F'\frac{\partial\eta}{\partial x} - F''C\frac{\partial\eta}{\partial y}, \quad B_{12} = [-F' + F''(B-A)]\frac{\partial\eta}{\partial y} - F''C\frac{\partial\eta}{\partial x},$$
$$A = \left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial x}\right)^2, \quad B = \left(\frac{\partial\xi}{\partial y}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2, \quad C = \frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}. \tag{7}$$

In accordance with [7], the characteristic determinant of system (6) is a fourth-degree polynomial

$$\Delta = \left(\sum_{k+l=3} A_{kl}\alpha^k\beta^l\right)\left(-\frac{\partial\xi}{\partial y}\alpha + \frac{\partial\xi}{\partial x}\beta\right) - \left(\sum_{k+l=3} B_{kl}\alpha^k\beta^l\right)\left(\frac{\partial\eta}{\partial y}\alpha - \frac{\partial\eta}{\partial x}\beta\right) =$$
$$= F'(\alpha^2+\beta^2)(B\alpha^2 - 2C\alpha\beta + A\beta^2) - F''[C(\alpha^2-\beta^2) + (B-A)\alpha\beta]^2. \tag{8}$$

As follows from (7) and incompressibility condition (6), the first two quantities from the group A, B, and C are positive and are connected with each other by the condition $A > 0$, $B > 0$, $AB - C^2 = 1$. As a result, the below quadratic form is positive-definite

$$B\alpha^2 - 2C\alpha\beta + A\beta^2 > 0. \tag{9}$$

It can be concluded on the basis of (8) and (9) that

$$\Delta > 0 \quad \text{at } F' > 0,\ F'' \leqslant 0,$$
$$\Delta < 0 \quad \text{at } F' < 0,\ F'' \geqslant 0. \tag{10}$$

With conditions for the elastic potential (10), the characteristic equation $\Delta = 0$ has no real roots. Thus, nonlinear system of equations (6) is elliptic for any of its solutions. As a result, (10) are sufficient conditions of ellipticity of the equations describing the plane strain of an incompressible elastic material.

3. When only stresses are assigned on the boundary of the region, the plane strain problem is conveniently formulated in stresses. Equations (1) for this problem appear as follows in complex coordinates of the strain state $z = x + iy$, $\bar{z} = x - iy$

$$\partial P^{11}/\partial z + \partial(P^{12} - 2V)/\partial\bar{z} = 0,$$
$$P^{11} = \overline{P^{22}} = -2F'\varepsilon^{11}, \quad P^{12} = 2F'(1-\varepsilon^{12}) - 2q,$$
$$\varepsilon^{11} = \overline{\varepsilon^{22}} = 2\frac{\partial u}{\partial\bar{z}}\left(1 - \frac{\partial\bar{u}}{\partial\bar{z}}\right), \quad 1 - \varepsilon^{12} = \left(1 - \frac{\partial u}{\partial z}\right)\left(1 - \frac{\partial\bar{u}}{\partial\bar{z}}\right) - \frac{\partial u}{\partial\bar{z}}\frac{\partial\bar{u}}{\partial z},$$
$$\varepsilon_1 - 2\varepsilon_2 = 0, \quad \varepsilon_1 = \varepsilon^{12}, \quad 4\varepsilon_2 = (\varepsilon^{12})^2 - \varepsilon^{11}\varepsilon^{22},$$
$$P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\bigg|_L = 2ip(s). \tag{11}$$

Equations (11) contain contravariant complex components of the vectors and tensors, which are connected with their Cartesian components by the formulas

$$u^1 = \overline{u^2} = u = u_x + iu_y, \quad p^1 = \overline{p^2} = p = p_x + ip_y,$$
$$P^{11} = \overline{P^{22}} = P_{xx} - P_{yy} + 2iP_{xy}, \quad P^{12} = P_{xx} + P_{yy},$$
$$\varepsilon^{11} = \overline{\varepsilon^{22}} = \varepsilon_{xx} - \varepsilon_{yy} + 2i\varepsilon_{xy}, \quad \varepsilon^{12} = \varepsilon_{xx} + \varepsilon_{yy}.$$

If we exclude the displacements from the expressions for the strains themselves, their first and second derivatives, and the incompressibility condition represented in the forms

$$(1-\varepsilon^{12})^2-\varepsilon^{11}\varepsilon^{22}=1,\quad \left(1-\frac{\partial u}{\partial z}\right)\left(1-\frac{\partial \bar{u}}{\partial \bar{z}}\right)-\frac{\partial u}{\partial \bar{z}}\frac{\partial \bar{u}}{\partial z}=1, \tag{12}$$

we obtain the strain compatibility condition in nonlinear elasticity for an incompressible material

$$\begin{gathered}4(1+|\varepsilon^{11}|^2)\operatorname{Re}\left(\bar{\varepsilon}^{11}\frac{\partial^2\varepsilon^{11}}{\partial z\,\partial\bar{z}}+\sqrt{1+|\varepsilon^{11}|^2}\,\frac{\partial^2\varepsilon^{11}}{\partial z^2}\right)=\\=2\operatorname{Re}\left[(\bar{\varepsilon}^{11})^2\frac{\partial\varepsilon^{11}}{\partial z}\frac{\partial\varepsilon^{11}}{\partial\bar{z}}\right]-\left|\frac{\partial\varepsilon^{11}}{\partial\bar{z}}\right|^2-(3+2|\varepsilon^{11}|^2)\left|\frac{\partial\varepsilon^{11}}{\partial z}\right|^2.\end{gathered} \tag{13}$$

Proceeding on the basis of the expressions for the principal invariants of the two-dimensional stress tensor and their combinations through the complex components of the stresses

$$P_1=P^{12},\ 4P_2=(P^{12})^2-P^{11}P^{22},\ I=P_1{}^2-4P_2=P^{11}P^{22} \tag{14}$$

we find that I will also be invariant. We can use (11) and (12) to establish its connection with the linear strain invariant and with hydrostatic pressure

$$I=4[F'(\varepsilon_1)]^2[(1-\varepsilon_1)^2-1],\ P^{12}+2q=\sqrt{4[F'(\varepsilon_1)]^2+I}.$$

It is easily seen that

$$\text{at}\ \ dI/d\varepsilon_1\neq 0\ \ \varepsilon_1=\varepsilon_1(I),\ 2q=\sqrt{4[F'(I)]^2+I}-P_1. \tag{15}$$

It follows from (11) and (15) that the strains are represented through the stresses by means of formulas that do not contain hydrostatic pressure:

$$\varepsilon^{11}=\overline{\varepsilon^{22}}=-P^{11}/2F'(I),\ 1-\varepsilon^{12}=\sqrt{1+I/[2F'(I)]^2}. \tag{16}$$

Insertion of (16) into (13) yields the stress compatibility equation in nonlinear elasticity for an incompressible material. Having augmented it by the equilibrium equation and boundary condition (11), we obtain the problem of the plane strain of an incompressible material in stresses and the complex coordinates of the strain state

$$\begin{gathered}4\left(1+\frac{|P^{11}|^2}{(2F')^2}\right)\operatorname{Re}\left(\frac{\overline{P^{11}}}{2F'}\frac{\partial^2}{\partial z\,\partial\bar{z}}\frac{P^{11}}{2F'}-\sqrt{1+\frac{|P^{11}|^2}{(2F')^2}}\frac{\partial^2}{\partial z^2}\frac{P^{11}}{2F'}\right)=\\=2\operatorname{Re}\left[\left(\frac{\overline{P^{11}}}{2F'}\right)^2\frac{\partial}{\partial z}\frac{P^{11}}{2F'}\frac{\partial}{\partial\bar{z}}\frac{P^{11}}{2F'}\right]-\left|\frac{\partial}{\partial\bar{z}}\frac{P^{11}}{2F'}\right|^2-\left(3+2\frac{|P^{11}|^2}{(2F')^2}\right)\left|\frac{\partial}{\partial z}\frac{P^{11}}{2F'}\right|^2,\\\frac{\partial P^{11}}{\partial z}+\frac{\partial(P^{12}-2V)}{\partial\bar{z}}=0,\quad P^{12}\frac{dz}{ds}-P^{11}\frac{d\bar{z}}{ds}\bigg|_L=2ip(s),\end{gathered} \tag{17}$$

where F'(I), while I is determined through the stress by Eq. (14).

If we introduce the Airy function U, assuming that

$$P^{11}=\overline{P^{22}}=-4\partial^2U/\partial\bar{z}^2,\ P^{12}=2V+4\partial^2U/\partial z\partial\bar{z}, \tag{18}$$

then we can represent (17) in the form of a boundary-value problem for this function: the function must satisfy a nonlinear fourth-order equation, while its first derivatives must take specified values on the boundary of the region

$$\begin{gathered}2S\operatorname{Re}\left[2\frac{1}{F'}\frac{\partial^2U}{\partial z^2}\frac{\partial^2}{\partial z\,\partial\bar{z}}\left(\frac{1}{F'}\frac{\partial^2U}{\partial\bar{z}^2}\right)+\sqrt{S}\frac{\partial^2}{\partial z^2}\left(\frac{1}{F'}\frac{\partial^2U}{\partial\bar{z}^2}\right)\right]=\\=8\operatorname{Re}\left[\left(\frac{1}{F'}\frac{\partial^2U}{\partial z^2}\right)^2\frac{\partial}{\partial z}\left(\frac{1}{F'}\frac{\partial^2U}{\partial\bar{z}^2}\right)\frac{\partial}{\partial\bar{z}}\left(\frac{1}{F'}\frac{\partial^2U}{\partial\bar{z}^2}\right)\right]-\left|\frac{\partial}{\partial z}\left(\frac{1}{F'}\frac{\partial^2U}{\partial z^2}\right)\right|^2-(1+2S)\left|\frac{\partial}{\partial z}\left(\frac{1}{F'}\frac{\partial^2U}{\partial\bar{z}^2}\right)\right|^2,\\2\frac{\partial U}{\partial\bar{z}}\bigg|_L=2\left(\frac{\partial U}{\partial\bar{z}}\right)_0+\int_0^s\left(ip-V\frac{dz}{ds}\right)ds,\\S=1+\left(\frac{2}{F'}\right)^2\left|\frac{\partial^2U}{\partial z^2}\right|^2,\quad F'=F'(I),\ I=16\left|\frac{\partial^2U}{\partial z^2}\right|^2.\end{gathered} \tag{19}$$

4. Let us examine a Mooney material [3], which to a certain extent reflects the behavior of incompressible rubber-like materials with finite strains and in the case of plane strain can be characterized by an elastic potential with one constant

$$F = -c\varepsilon_1,\ c = \text{const} > 0,\ F' = -c,\ F'' = 0. \tag{20}$$

For the Mooney material, problem (19) will be

$$2S_0 \operatorname{Re}\left(2\frac{\partial^2 U}{\partial z^2}\frac{\partial^4 U}{\partial z\,\partial \bar z^3} - \sqrt{S_0}\frac{\partial^4 U}{\partial z^2\,\partial \bar z^2}\right) = 8\operatorname{Re}\left[\left(\frac{\partial^2 U}{\partial z^2}\right)^2\frac{\partial^3 U}{\partial z\,\partial \bar z^2}\frac{\partial^3 U}{\partial \bar z^3}\right] - c^2\left|\frac{\partial^3 U}{\partial z^3}\right|^2 - (1+2S_0)\left|\frac{\partial^3 U}{\partial z\,\partial \bar z^2}\right|^2,$$

$$S_0 = c^2 + 4\left|\frac{\partial^2 U}{\partial z^2}\right|^2,\quad 2\frac{\partial U}{\partial \bar z}\bigg|_L = 2\left(\frac{\partial U}{\partial \bar z}\right)_0 + \int_0^s\left(ip - V\frac{dz}{ds}\right)ds \equiv t(s). \tag{21}$$

We will study the form and type of equation for the Airy function with applied loads of different intensities. We will use P_0 and $\sigma = P_0/c$ to denote the characteristic load and dimensionless stress and we will set $U_* = U/P_0$. Introducing these quantities in (21), we obtain

$$2S_* \operatorname{Re}\left(2\sigma\frac{\partial^2 U_*}{\partial z^2}\frac{\partial^4 U_*}{\partial z\,\partial \bar z^3} - \sqrt{S_*}\frac{\partial^4 U_*}{\partial z^2\,\partial \bar z^2}\right) = -(1+2S_*)\left|\frac{\partial^3 U_*}{\partial z\,\partial \bar z^2}\right|^2 +$$

$$+ 8\sigma^3\operatorname{Re}\left[\left(\frac{\partial^2 U_*}{\partial z^2}\right)^2\frac{\partial^3 U_*}{\partial z\,\partial \bar z^2}\frac{\partial^3 U_*}{\partial \bar z^3}\right] - \sigma\left|\frac{\partial^3 U_*}{\partial z^3}\right|^2,\quad S_* = 1 + 4\sigma^2\left|\frac{\partial^2 U_*}{\partial z^2}\right|^2. \tag{22}$$

We will distinguish the following loading regimes: light loading at $P_0 \ll c$ ($\sigma \ll 1$), moderate loading at $P_0 \sim c$ ($\sigma \sim 1$), and heavy loading at $P_0 \gg c$ ($\sigma \gg 1$).

With moderate loading, all of the terms in (22) are roughly of the same order. Thus, all of them should be retained. This equation is a general nonlinear model. Equation (22) is of the elliptic type, as is the corresponding system of equations for the initial coordinates. By virtue of (20), ellipticity conditions (10) are satisfied for the latter system. This result can also be established directly. In fact, representing (22) in the real variables x, y

$$\Phi \equiv 32\operatorname{Re}\left(\sqrt{1 + 4\sigma^2\left|\frac{\partial^2 U_*}{\partial z^2}\right|^2}\frac{\partial^4 U_*}{\partial z^2\,\partial \bar z^2} - 2\sigma\frac{\partial^2 U_*}{\partial z^2}\frac{\partial^4 U_*}{\partial z\,\partial \bar z^3}\right) + H =$$

$$= (E-M)\frac{\partial^4 U_*}{\partial x^4} - 2N\frac{\partial^4 U_*}{\partial x^3\,\partial y} + 2E\frac{\partial^4 U_*}{\partial x^2\partial y^2} - 2N\frac{\partial^4 U_*}{\partial x\,\partial y^3} + (E+M)\frac{\partial^4 U_*}{\partial y^4} + H = 0,$$

$$E = \sqrt{1 + M^2 + N^2},\quad M = \frac{\sigma}{2}\left(\frac{\partial^2 U_*}{\partial x^2} - \frac{\partial^2 U_*}{\partial y^2}\right),\quad N = \sigma\frac{\partial^2 U_*}{\partial x\,\partial y}$$

(where H represents terms with lower derivatives), we find that its characteristic equation for any solution has only complex roots

$$\Delta_* = \sum_{m+n=4}\left(\partial\Phi/\partial\frac{\partial^4 U_*}{\partial x^m\,\partial y^n}\right)\alpha^m\beta^n = (1+k^2)[(E-M)k^2 + 2Nk + E + M] = 0,$$

$$k = -\alpha/\beta,\ k_{1,2} = \pm i,\ k_{3,4} = (-N \pm i)/(E-M). \tag{23}$$

With light loading, the terms in (22) which contain the dimensionless stress will be small compared to the other terms. Ignoring these terms leads us to a biharmonic equation which establishes a linear model of elasticity

$$\Delta\Delta U_* = 16\partial^4 U_*/(\partial z^2\partial \bar z^2). \tag{24}$$

As (22), Eq. (24) is of the elliptic type: the corresponding characteristic equation [which follows from (23) at E = 1, M = N = 0] has purely imaginary roots

$$\Delta_* = (1 + k^2)^2 = 0,\quad k_{1,2} = \pm i,\quad k_{3,4} = \pm i.$$

Thus, the linear theory of elasticity follows from the nonlinear theory of elasticity for a Mooney material with characteristic loads which are small compared to the elastic constant of the material.

Finally, with heavy loading, the terms in (22) that contain the dimensionless stress will now be large relative to the other terms. Leaving only the largest terms in the equation, we obtain a new nonlinear fourth-order equation which establishes a special nonlinear model of elasticity:

$$2\left|\frac{\partial^2 U_*}{\partial z^2}\right|^2 \operatorname{Re}\left(\frac{\partial^2 U_*}{\partial z^2}\frac{\partial^4 U_*}{\partial z\,\partial \bar{z}^3} - \left|\frac{\partial^2 U_*}{\partial z^2}\right|\frac{\partial^4 U_*}{\partial z^2\,\partial \bar{z}^2}\right) = \operatorname{Re}\left[\left(\frac{\partial^2 U_*}{\partial z^2}\right)^2 \frac{\partial^3 U_*}{\partial z\,\partial \bar{z}^2}\frac{\partial^3 U_*}{\partial \bar{z}^3}\right] - \left|\frac{\partial^2 U_*}{\partial z^2}\right|^2 \left|\frac{\partial^3 U_*}{\partial z\,\partial \bar{z}^2}\right|^2. \tag{25}$$

In contrast to (22) and (24), Eq. (25) is of the compound type: its characteristic equation [which follows from (23) after the small terms are ignored] has two imaginary and two coincident real roots

$$\Delta_* = (1+k^2)\left[(E_0 - M_0)k^2 + 2N_0 k + E_0 + M_0\right] = 0,$$
$$k_{1,2} = \pm i,\ k_{3,4} = -N_0/(E_0 - M_0),$$
$$E_0 = \sqrt{M_0^2 + N_0^2},\quad 2M_0 = \partial^2 U_*/\partial x^2 - \partial^2 U_*/\partial y^2,\quad N_0 = \partial^2 U_*/(\partial x\,\partial y).$$

The change in the type of equation for the Airy function at loads which are considerably greater than the elastic constant reflects the change in the mechanical properties of the material under these conditions.

Let us return to the general case and change over in Eq. (21) from the variables z, $\bar{z}$ to other complex variables τ, $\bar{\tau}$ by means of the analytic function τ = τ(z), which is equivalent to changing over from the Cartesian coordinates x, y (z = x + iy) to orthogonal curvilinear coordinates λ, μ (τ = λ + iμ) in the same plane. The following representations are valid for the determinant of the transform and the derivative of the logarithm of the determinant

$$Q(\tau,\bar{\tau}) = \frac{\partial(\tau,\bar{\tau})}{\partial(z,\bar{z})} = \tau_z \bar{\tau}_{\bar{z}} = \left(z_\tau \bar{z}_{\bar{\tau}}\right)^{-1},\quad A(\tau) = \frac{\partial \ln Q}{\partial \tau} = -\frac{z_{\tau\tau}}{z_\tau}. \tag{26}$$

Transformed problem (21) for the function U(τ, $\bar{\tau}$) will be written in the form

$$\begin{gathered}
2S\left\{2\operatorname{Re}\left[\left(\frac{\partial^2 U}{\partial \bar{\tau}^2} + \bar{A}\frac{\partial U}{\partial \bar{\tau}}\right)\left(\frac{\partial^4 U}{\partial \tau^3\,\partial \bar{\tau}} + 3A\frac{\partial^3 U}{\partial \tau^2\,\partial \bar{\tau}} + (A' + 2A^2)\frac{\partial^2 U}{\partial \tau\,\partial \bar{\tau}}\right)\right] + \right.\\
\left. + \left|\frac{\partial^3 U}{\partial \tau^2\,\partial \bar{\tau}} + A\frac{\partial^2 U}{\partial \tau\,\partial \bar{\tau}}\right|^2 - \sqrt{S}\left(\frac{\partial^4 U}{\partial \tau^2\,\partial \bar{\tau}^2} + A\frac{\partial^3 U}{\partial \tau\,\partial \bar{\tau}^2} + \bar{A}\frac{\partial^3 U}{\partial \tau^2\,\partial \bar{\tau}} + A\bar{A}\frac{\partial^2 U}{\partial \tau\,\partial \bar{\tau}}\right)\right\} = \\
= 8\operatorname{Re}\left\{\left(\frac{\partial^2 U}{\partial \bar{\tau}^2} + \bar{A}\frac{\partial U}{\partial \bar{\tau}}\right)^2\left(\frac{\partial^3 U}{\partial \tau^2\,\partial \bar{\tau}} + A\frac{\partial^2 U}{\partial \tau\,\partial \bar{\tau}}\right)\left(\frac{\partial^3 U}{\partial \tau^3} + 3A\frac{\partial^2 U}{\partial \tau^2} + (A' + 2A^2)\frac{\partial U}{\partial \tau}\right)\right\} - \\
- \frac{c^2}{Q^2}\left\{\left|\frac{\partial^3 U}{\partial \tau^3} + 3A\frac{\partial^2 U}{\partial \tau^2} + (A' + 2A^2)\frac{\partial U}{\partial \tau}\right|^2 + \left|\frac{\partial^3 U}{\partial \tau^2\,\partial \bar{\tau}} + A\frac{\partial^2 U}{\partial \tau\,\partial \bar{\tau}}\right|^2\right\}, \\
S = \frac{c^2}{Q^2} + 4\left|\frac{\partial^2 U}{\partial \tau^2} + A\frac{\partial U}{\partial \tau}\right|^2,\quad 2\frac{\partial U}{\partial \bar{\tau}}\bigg|_{L'} = t\frac{d\bar{z}}{d\bar{\tau}}\bigg|_{L'}
\end{gathered} \tag{27}$$

(L' is the transformed contour L).

Due to (18), the physical components of the stresses in the coordinates λ, μ are expressed through the solutions of Eq. (27) by means of the formulas [8]

$$\begin{gathered}
P_{\lambda\lambda} - P_{\mu\mu} + 2iP_{\lambda\mu} = \frac{\bar{z}_{\bar{\tau}}}{z_\tau}P^{11} = -4Q\left(\frac{\partial^2 U}{\partial \bar{\tau}^2} + \bar{A}\frac{\partial U}{\partial \bar{\tau}}\right), \\
P_{\lambda\lambda} + P_{\mu\mu} = P^{12} = 2V + 4Q\frac{\partial^2 U}{\partial \tau\,\partial \bar{\tau}}.
\end{gathered} \tag{28}$$

This equation for the Airy function in the form (27) is convenient to use to find classes of exact solutions by selecting the coordinate transformation and finding U as a function (for example) of a single real variable (such as λ or μ) or their combination. We then use the resulting Airy function to determine the stress field, which can in turn be employed to solve certain boundary-value problems with boundary conditions of a special form. Let us look at some examples.

5. We assume that the quantity A which figures into (27) is constant and that the corresponding coordinate transformation and its determinant have the expressions

$$z = \exp(-A\tau),\ A = \text{const},\ Q = (A\bar{A})^{-1}\exp(A\tau + \bar{A}\bar{\tau}).$$

It follows from the relations

$$z = r\exp(i\theta),\ \tau = \lambda + i\mu,\ A = a + ib,$$
$$\lambda = -\frac{b\theta + a\ln r}{a^2 + b^2},\quad \mu = -\frac{a\theta - b\ln r}{a^2 + b^2} \tag{29}$$

that the curves λ = const and μ = const form orthogonal families of logarithmic spirals.

Assuming that body forces are absent (V = 0), we seek the solution of Eq. (27) in the form

$$U = U(\xi),\ \xi = A\tau + \bar{A}\bar{\tau} = 2(a\lambda - b\mu),$$

so that the Airy function satisfies the equation

$$2S\{2(U'' + U')(U^{IV} + 3U''' + 2U'') + (U''' + U'')^2 -$$
$$-\sqrt{S}(U^{IV} + 2U''' + U'')\} = 8(U'' + U')^2(U''' + U'')(U''' + \tag{30}$$
$$+ 3U'' + 2U') - c^2\exp(-2\xi)\{(U''' + 3U'' + 2U')^2 + (U''' + U'')^2\},\ S = c^2\exp(-2\xi) + 4(U'' + U')^2,$$

while the components of the stresses (28) are determined by the expressions

$$P_{\lambda\lambda} - P_{\mu\mu} + 2iP_{\lambda\mu} = -4(\exp\xi)(U'' + U')\bar{A}^2/|A|^2,$$
$$P_{\lambda\lambda} + P_{\mu\mu} = 4(\exp\xi)U''. \tag{31}$$

Equation (30) can be completely integrated. In fact, by substituting the function

$$2(U'' + U') = c\exp(-\xi)\operatorname{sh} w \tag{32}$$

we reduce it to the equation

$$w'' = w'(w' + 1)$$

and after integration it yields $\exp(-w) = h + g\exp\xi$, h = const, g = const. Returning to (32) and integrating, we obtain

$$4(\exp\xi)(U'' + U') = c[(h + g\exp\xi)^{-1} - (h + g\exp\xi)],$$
$$4(\exp\xi)U' = c\{(h^{-1} - h)\xi - h^{-1}\ln|h + g\exp\xi| - (h + g\exp\xi) - f\},\ f = \text{const}. \tag{33}$$

Equations (31) and (33) determine the stress field, which is dependent on the free parameters g, h, f:

$$P_{\lambda\lambda} = \frac{c}{2}\Big\{f + (h - h^{-1})\xi + h^{-1}\ln|h + g\exp\xi| + \frac{1-m}{h + g\exp\xi} + m(h + g\exp\xi)\Big\},$$
$$P_{\mu\mu} = \frac{c}{2}\Big\{f + (h - h^{-1})\xi + h^{-1}\ln|h + g\exp\xi| + \frac{1+m}{h + g\exp\xi} - m(h + g\exp\xi)\Big\}, \tag{34}$$
$$P_{\lambda\mu} = nc\{(h + g\exp\xi)^{-1} - (h + g\exp\xi)\},$$
$$m = \frac{a^2 - b^2}{a^2 + b^2},\quad n = \frac{ab}{a^2 + b^2},\quad \xi = 2(a\lambda - b\mu).$$

The stress field which is found can be used to solve a number of boundary value problems. For example, we will examine a curvilinear tetragon whose sides are segments of spirals (29):

$$\lambda = \lambda_-,\ \lambda = \lambda_+;\ \mu = \mu_-,\ \mu = \mu_+. \tag{35}$$

The boundary stresses on the sides of the tetragon, corresponding to stresses (34), have variable normal and tangential components determined by the expressions

$$\begin{gathered}
\lambda = \lambda_\pm,\ \xi_\pm = 2(a\lambda_\pm - b\mu),\\
(p_\lambda)_\pm = \frac{c}{2}\Big\{f + \left(h - h^{-1}\right)\xi_\pm + h^{-1}\ln|h + g\exp\xi_\pm| +\\
+ \frac{1-m}{h+g\exp\xi_\pm} + m(h + g\exp\xi_\pm)\Big\},\\
(p_\mu)_\pm = nc\{(h + g\exp\xi_\pm)^{-1} - (h + g\exp\xi_\mp)\};\\
\mu = \mu_\pm,\ \xi'_\pm = 2(a\lambda - b\mu_\pm),\\
(p_\mu)_\pm = \frac{c}{2}\Big\{f + \left(h - h^{-1}\right)\xi'_\pm + h^{-1}\ln|h + g\exp\xi'_\pm| +\\
+ \frac{1+m}{h+g\exp\xi'_\pm} - m\left(h + g\exp\xi'_\pm\right)\Big\},\\
(p_\lambda)_\pm = nc\{(h + g\exp\xi'_\pm)^{-1} - (h + g\exp\xi'_\pm)\}.
\end{gathered} \tag{36}$$

Equations (34) give the exact solution of the problem of the equilibrium of a tetragon (35) with boundary conditions (36).

In particular, with b = 0, Eqs. (29) will be $\lambda = -(\ln r)/a$, $\mu = -\theta/a$. Here, the curves λ = const and μ = const form families of concentric circles and rays. In Eqs. (34), m = 1, n = 0, and the stresses depend only on one coordinate:

$$\begin{gathered}
P_{\lambda\lambda} = (c/2)\{f + (h - h^{-1})\xi + h^{-1}\ln|h + g\exp\xi| + h + g\exp\xi\},\\
P_{\mu\mu} = (c/2)\{f + (h - h^{-1})\xi + h^{-1}\ln|h + g\exp\xi| + 2(h + g\exp\xi)^{-1} - h - g\exp\xi\},\\
P_{\lambda\mu} = 0,\ \xi = 2a\lambda.
\end{gathered} \tag{37}$$

In this case, tetragon (35) is bounded by segments of circles and rays and is loaded only by normal loads which are constant on the curved sections and variable on the straight sections:

$$\begin{gathered}
\lambda = \lambda_\pm,\ \xi_\pm = 2a\lambda_\pm,\ (p_\mu)_\pm = 0,\\
(p_\lambda)_\pm = (c/2)\{f + (h - h^{-1})\xi_\pm + h^{-1}\ln|h + g\exp\xi_\pm| + h + g\exp\xi_\pm\};\\
\mu = \mu_\pm,\ \xi = 2a\lambda,\ (p_\lambda)_\pm = 0,\\
(p_\mu)_\pm = (c/2)\{f + (h - h^{-1})\xi + h^{-1}\ln|h + g\exp\xi| + 2(h + g\exp\xi)^{-1} - h - g\exp\xi\}.
\end{gathered} \tag{38}$$

Equations (37) solve the problem of the equilibrium of a part of a circular ring bounded by two radii with boundary loads (38).

6. Let the coordinate transformation have the form

$$\begin{gathered}
z = a\operatorname{ch}\tau,\ a = \bar{a} = \text{const},\ z = x + iy,\ \tau = \lambda + i\mu,\\
x = a\operatorname{ch}\lambda\cos\mu,\ y = a\operatorname{sh}\lambda\sin\mu,\\
x^2/(a\operatorname{ch}\lambda)^2 + y^2/(a\operatorname{sh}\lambda)^2 = 1,\ x^2/(a\cos\mu)^2 - y^2/(a\sin\mu)^2 = 1.
\end{gathered}$$

It is clear from this that the curves λ = const and μ = const form orthogonal families of ellipses and hyperbolas. The quantities (26) are equal to

$$Q = 1/(a^2\operatorname{sh}\tau\operatorname{sh}\bar{\tau}),\ A = -\operatorname{ch}\tau/\operatorname{sh}\tau.$$

We seek a solution of Eq. (27) of the type

$$U = U(\eta),\ \eta = \operatorname{ch}\tau + \operatorname{ch}\bar{\tau} = 2\operatorname{ch}\lambda\cos\mu,$$

the equation and the stresses (28) in the absence of body forces accordingly being

$$[c^2a^4 + 4(U'')^2]U^{\mathrm{IV}} = 2(U''')^2(2U'' + \sqrt{c^2a^4 + 4(U'')^2}),$$
$$P_{\lambda\lambda} - P_{\mu\mu} + 2iP_{\lambda\mu} = -\frac{4U''}{a^2}\frac{\operatorname{sh}^2\bar{\tau}}{\operatorname{sh}\tau\operatorname{sh}\bar{\tau}},\quad P_{\lambda\lambda} + P_{\mu\mu} = \frac{4U''}{a^2}. \tag{39}$$

Integration of this equation twice yields

$$4U'' = ca^2[g\eta + h - (g\eta + h)^{-1}],$$
$$g = \text{const},\ h = \text{const},\ \eta = 2\operatorname{ch}\lambda\cos\mu. \tag{40}$$

Equations (39) and (40) determine stress fields containing the free parameters g and h:

$$P_{\lambda\lambda} = \frac{c}{2}[g\eta + h - (g\eta + h)^{-1}]\frac{(1 + \operatorname{ch}2\lambda)(1 - \cos 2\mu)}{\operatorname{ch}2\lambda - \cos 2\mu},$$
$$P_{\mu\mu} = -\frac{c}{2}[g\eta + h - (g\eta + h)^{-1}]\frac{(1 - \operatorname{ch}2\lambda)(1 + \cos 2\mu)}{\operatorname{ch}2\lambda - \cos 2\mu}, \tag{41}$$
$$P_{\lambda\mu} = \frac{c}{2}[g\eta + h - (g\eta + h)^{-1}]\frac{\operatorname{sh}2\lambda\sin 2\mu}{\operatorname{ch}2\lambda - \cos 2\mu}.$$

In particular, the stresses at the boundaries of an elliptical ring $\lambda_- \le \lambda \le \lambda_+$ have the values

$$\text{at } \lambda = \lambda_\pm\ \ \eta_\pm = 2\operatorname{ch}\lambda_\pm\cos\mu,$$
$$(p_\lambda)_\pm = \frac{c}{2}[g\eta_\pm + h - (g\eta_\pm + h)^{-1}]\frac{(1 + \operatorname{ch}2\lambda_\pm)(1 - \cos 2\mu)}{\operatorname{ch}2\lambda_\pm - \cos 2\mu}, \tag{42}$$
$$(p_\mu)_\pm = \frac{c}{2}[g\eta_\pm + h - (g\eta_\pm + h)^{-1}]\frac{\operatorname{sh}2\lambda_\pm\sin 2\mu}{\operatorname{ch}2\lambda_\pm - \cos 2\mu},$$

Here, the breaking force on the boundary contours is determined by the expressions

$$(P_{\mu\mu})_\pm = -\frac{c}{2}[g\eta_\pm + h - (g\eta_\pm + h)^{-1}]\frac{(1 - \operatorname{ch}2\lambda_\pm)(1 + \cos 2\mu)}{\operatorname{ch}2\lambda_\pm - \cos 2\mu}. \tag{43}$$

Equations (41) and (42) give the solution of the problem of the equilibrium of an elliptical ring subjected to periodic contour loads. In particular, at $\lambda_- = 0$, the internal contour becomes a segment of the x-axis $-a \le x \le a$, while the problem in question becomes the problem of the loading of an elliptical region with a straight slit. The boundary conditions on the slit are simplified:

$$\lambda_- = 0,\ (p_\lambda)_-^0 = c[2g\cos\mu + h - (2g\cos\mu + h)^{-1}],\ (p_\mu)_-^0 = 0,$$

i.e., the slit is loaded only by a normal load. The breaking force (43) on the slit takes the values

$$\text{at }\ \mu \ne 0;\ \pi\ \ \lambda_- = 0,\ (P_{\mu\mu})_-^0 = 0,$$
$$\text{at }\ \mu = 0\ \ \lambda_- = 0,\ (P_{\mu\mu})_-^0 = c[h + 2g - (h + 2g)^{-1}], \tag{44}$$
$$\text{at }\ \mu = \pi\ \ \lambda_- = 0,\ (P_{\mu\mu})_-^0 = c[h - 2g - (h - 2g)^{-1}],$$

i.e., it is equal to zero on the entire slit except for its ends and is finite and different at the ends of the slit (at $h \pm 2g \ne 0$). It is easy to see that at g = 0 the contour load (42) is symmetrical (it is constant on the slit). Accordingly, the breaking forces (44) at the ends of the slit are equal and finite.

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ANALYSIS OF FRINGE PATTERNS BY THE METHOD OF INTEGRAL BOUNDARY EQUATIONS IN THE SOLUTION OF PLANE ELASTOPLASTIC PROBLEMS

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In experimental studies of plane problems of the mechanics of deformable bodies by moire methods [1-3] or holographic interferometry with the use of superimposed interferometers [4], the information that is obtained is represented in the form of patterns of interference fringes. By analyzing these patterns, it is possible to determine the stress and strain fields in the region being studied. There are various approaches and corresponding algorithms for solving problems [2, 5-9, etc.] based on determination of fringe-order functions N(x, y) in the region being studied, the transition from these functions to functions of the plane components of the displacements u(x, y) and v(x, y), and determination of their partial derivatives.

The fact that the strain components are calculated by differentiating reconstructed functions makes these methods highly sensitive to errors and distortions in the initial data and to the choice for the criterion of their approximation. At the same time, the information obtained from the experiment is inadequate to correctly approximate the initial functions, since it is necessary to know not only the orders of the fringes at the boundaries of the region but also their derivatives. Application packages currently available for analyzing fringe patterns [9-12] automatically sample and numerically filter the initial data, which reduces the laboriousness of the calculations considerably. However, the algorithms used for subsequent analysis still have the deficiencies noted above.

The authors of [13] noted the efficacy of synthesizing holographic interferometry and numerical potential methods to study the elastoplastic state of three-dimensional bodies. Here, to establish the stress—strain state inside the region, it is sufficient to have information that can be obtained from the fringe patterns at its boundaries. Among the advantages of this approach is the smoothing effect inherent in integral methods: the errors of the boundary conditions turn out to be considerably lower farther into the region than near the boundaries.

In the present study, we examine the feasibility of using theoretical solutions obtained by numerical realization of the method of integral boundary equations (IBE) to analyze fringe patterns in an investigation of elastoplastic fields of stress and strain.

1. Formulation of the Problem. Four fringe patterns are recorded [4] to find the plane components of the displacements u(x, y) and v(x, y) with the use of superimposed interferometers. In this case, the values of the displacements can be found from the formulas

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